Investigating Non-linear Asymptotes and the Graphs of Rational Functions

Marvin Harrell, *Emporia State University*
Dawn Slavens, *Midwestern State University*
Connie Richardson, *The University of Texas at Austin*

Abstract: *When looking at rational functions and their associated asymptotes, we often only discuss vertical, horizontal, and oblique asymptotes with our students. Often students ask if there are other types of asymptotes. This calculator-based activity allows students to investigate this question. In this activity, students will investigate the end behaviors of rational functions and how they are directly related to the end behaviors of their associated quotient polynomials.*

Keywords. *Rational function, asymptotes, multiple representations*

1 Introduction

This activity is a companion to two activities previously published in the *Ohio Journal*, namely "Investigating vertical asymptotes and the graphs of rational functions" (Harrell & Slavens, 2004) and "Investigating horizontal asymptotes and the graphs of rational functions" (Slavens & Harrell, 2006). The authors recommend that the students first complete the activity found in the Autumn 2006 issue or a similar activity investigating horizontal asymptotes; however, this is not a necessary requirement (*Editors' note: download links to both articles are provided in the list of references at the end of this article*). The activity that follows assumes that students have access to a graphing calculator and have an understanding of polynomials, functions and their graphs, and know how to divide polynomials. Students should record the majority of their work on a separate sheet of paper. The activity with additional space for student work, a PowerPoint file with a review and the answer key may be found at `http://faculty.mwsu.edu/math/dawn.slavens`.

2 Oblique and Non-linear Asymptote Activity

2.1 Overview

In this activity you will investigate the end behavior of rational functions. A rational function is one that can be written as $y = f(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials. First, recall how rational numbers can be rewritten. Consider, for instance, the rational number $\frac{141}{8}$ with dividend 141 and divisor 8. One can use long division to find the quotient q and remainder r satisfying the equation $141 = 8 \cdot q + r$, with $0 \leq r < 8$. Is $\frac{141}{8}$ equal to $q + \frac{r}{d}$? Verify your answer. Is this true for all rational numbers?

2.2 Extending to Rational Functions

1. Let's investigate a rational function where the degree of the numerator is one greater than the degree of the denominator. For example, we can examine the rational function $y = \frac{4x^2-4x-3}{x+3}$.
   a) Use either long division or synthetic division to determine the quotient and remainder. (Hint: The remainder should be 45. If you don't get 45, raise your hand.)

      First, identify the numerator & denominator of the rational function:

      i. $N(x) =$ ____________________

      ii. $D(x) =$ ____________________.

      Next, determine the quotient and remainder:

      i. Quotient = $Q(x) =$ ____________________

      ii. Remainder = $R(x) = 45$

      Rational functions of the form $y = f(x) = \frac{N(x)}{D(x)}$ can be written as $y = f(x) = \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, where $Q(x)$ and $R(x)$ are the quotient polynomial and the remainder polynomial, respectively, when we divide $N(x)$ by $D(x)$.

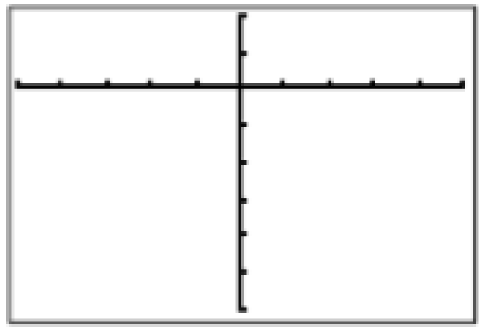
      Rewrite $y = \frac{4x^2-4x-3}{x+3}$ in the form $Q(x) + \frac{R(x)}{D(x)}$: ____________________.

   b) Let's examine the y values of the rational function $y = \frac{4x^2-4x-3}{x+3}$ and the y values of the quotient function $y = 4x - 16$. Use your calculator to generate a table of y values for both of these functions for the x values in the table below. For example, on the TI-84, one would put the rational function in y_1 and the quotient function in y_2. Next, set up your table by accessing [TBLSET] and set the "Indpnt" variable to "Ask" so that you can input your x values into the table. Finally, select and enter the x values from the tables below, recording the corresponding y values.

x	-100	-50	-40	-30	-20	-10		10	20	30	40	50	100
y_1													
y_2													

What does this seem to suggest about the behavior of the two functions, y_1 and y_2, as $x \to \pm\infty$?

c) Graph the rational function $y = \frac{4x^2-4x-3}{x+3}$ and the quotient function $y = 4x - 16$ using your graphing calculator with the following window settings: $x : [-25, 25]$, scale: 5; $y : [-150, 50]$, scale: 25. Using two different colors, sketch the graphs of both below.



d) Where do the graphs of the rational function and the quotient function look alike? Where do they look different?

e) Notice that as $x \to \pm\infty$ the graph of $y = \frac{4x^2-4x-3}{x+3}$ approaches the graph of the quotient function $y = 4x - 16$. Since the quotient function's degree is one, its graph is a line. When the graph of a rational function approaches a non-horizontal line as $x \to \pm\infty$, that line is called an **oblique asymptote** of the rational function. For our rational function, $y = 4x - 16$ is an equation of the oblique asymptote. Knowing that $y = \frac{4x^2-4x-3}{x+3}$ can be expressed as $y = 4x - 16 + \frac{45}{x+3}$, explain why the graph of $y = \frac{4x^2-4x-3}{x+3}$ must approach the line $y = 4x - 16$ as $x \to \pm\infty$. (Hint: Think about what happens to $\frac{45}{x+3}$ as $x \to \pm\infty$.)

f) In part (e) above, you explained why the graphs are "close" to each other as $x \to \pm\infty$. Give one possible reason why the graphs are not "close" to each other near $x = -3$.

g) In part (e) above, you learned that the rational function $y = \frac{4x^2-4x-3}{x+3}$ has an oblique asymptote that was obtained by dividing the numerator of the rational function by the denominator of the rational function. Also, you learned that an oblique asymptote of a rational function is a non-horizontal line, so it is defined by a degree 1 polynomial. What must the relationship between the degrees of the numerator and the denominator of a rational function be in order for the rational function to have an oblique asymptote? Explain.

2. Now let's consider the rational function $y = \frac{2x^3-7x^2-13x+18}{x+1}$ where the degree of the numerator is 2 greater than the degree of the denominator.

   a) Use long division or synthetic division to determine the quotient polynomial, $Q(x)$, and the remainder polynomial, $R(x)$. Then write the function in the form of $Q(x) + \frac{R(x)}{D(x)}$.

      i. Quotient = $Q(x) =$ ____________________

      ii. Remainder = $R(x) = 22$

      iii. $Q(x) + \frac{R(x)}{D(x)} =$ ____________________

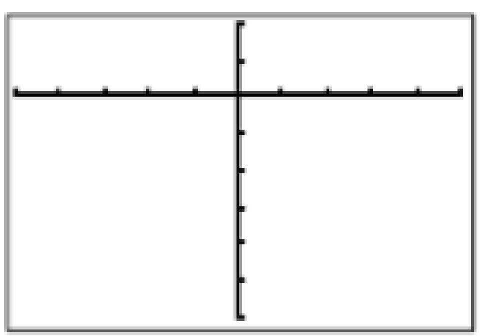
   How does the degree of this quotient polynomial differ from the degree of the quotient polynomial in problem 1(a)? What could be the reason for this difference?

   b) As in 1(b) examine the y values of the rational function and the quotient function $y = Q(x)$. Remember to let y_1 represent the rational function and y_2 the quotient function.

x	-42	-34	-26	-18	-10	-2	0	2	10	18	26	34	42
y_1													
y_2													

   What appears to be true about the values of the rational function relative to the values of the quotient function as $x \to \pm\infty$?

   c) Graph the rational function and the quotient function using your graphing calculator with the following window settings: $x : [-10, 10]$, scale: 1; $y : [-40, 100]$, scale: 20. Using two different colors, sketch the graphs of both below.



   d) Notice that as $x \to \pm\infty$, the graph of $y = \frac{2x^3-7x^2-13x+18}{x+1}$ approaches the graph of the quotient function $y = Q(x)$. This quotient function is **non-linear** because the quotient polynomial's degree is not 1. Knowing that $y = \frac{2x^3-7x^2-13x+18}{x+1}$ can be expressed as $y = Q(x) + \frac{22}{x+1}$, explain why the graph of $y = \frac{2x^3-7x^2-13x+18}{x+1}$ must approach the graph of $y = Q(x)$ as $x \to \pm\infty$. (Hint: If needed, refer to the hint in 1(e).)

3. Consider the rational function $y = \frac{x^5 - 5x^3 + 4x}{x^2 + x - 12}$.

   a) Find the following:

      i. Quotient = $Q(x) =$ ____________________

      ii. Remainder = $R(x) = 120x - 240$

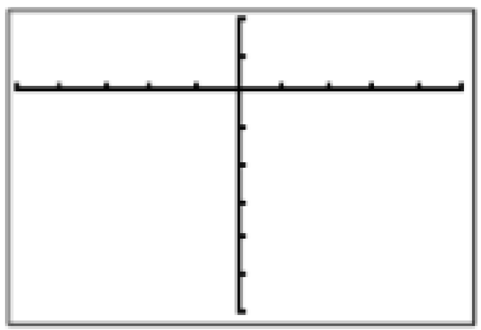
      iii. $Q(x) + \frac{R(x)}{D(x)} =$ ____________________

   The degree of the numerator is how much greater than the degree of the denominator?

   What is the degree of the quotient polynomial?

   Your answers to the previous two questions should be the same. Why should they be the same?

   b) Graph the rational function and the quotient function using your graphing calculator with the following window settings: $x : [-8, 8]$, scale: 1; $y : [-500, 500]$, scale: 100. Sketch the graphs of both below.



   c) What do the graphs in part (b) suggest to you about the rational function and its corresponding quotient function? Explain.

4. Consider the rational function $y = \frac{-2x^5 - 58x^3 - 200x}{x-1} = -2x^4 - 2x^3 + 56x^2 + 56x - 144 - \frac{144}{x-1}$.

   a) Use the above to determine the non-linear asymptote of the graph of $y = \frac{-2x^5 - 58x^3 - 200x}{x-1}$ as $x \to \pm\infty$. (Note that the division has already been done for you.)

   b) Verify your answer by graphing the rational function and your non-linear asymptote using the window $x : [-6, 6]$, scale:1; $y : [-200, 400]$, scale: 50.

5. Find the equation of the oblique and non-linear asymptote as $x \to \pm\infty$ for each of the following rational functions. Graph each rational function and asymptote in a suitable window to verify your results.

   a) $y = \frac{x^4-x^3+x^2-x+2}{x-2}$

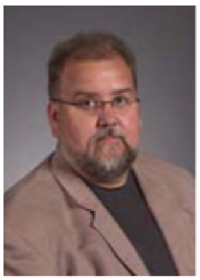
   b) $y = \frac{x^2-x}{x+2}$

   c) $y = \frac{x^3-2x^2+4x-9}{2x-3}$

References

Harrell & Slavens (Autumn 2004). Investigating vertical asymptotes and the graphs of rational functions. *Ohio Journal of School Mathematics 50,* 64-70. Available on-line at https://www.dropbox.com/s/zfki4i74585ih4a/Article_2004.pdf.

Slavens & Harrell (Autumn 2006). Investigating horizontal asymptotes and the graphs of rational functions. *Ohio Journal of School Mathematics 52,* 38-45. Available on-line at https://www.dropbox.com/s/zfki4i74585ih4a/Article_2004.pdf.



Marvin Harrell, mharrell@emporia.edu, is a professor of mathematics at Emporia State University. His major areas of interests include geometry, developing inquiry-based activities for use in grades 5-12, and teacher training.

Dawn Slavens, dawn.slavens@mwsu.edu, is a professor of mathematics at Midwestern State University and is chair of its Mathematics Department. She has taught mathematics at the middle and high school levels. Her interests include geometry, analysis, and teacher training.

Connie Richardson, cjrichardson@austin.utexas.edu, is a program coordinator at the Charles A. Dana Center at the University of Texas at Austin. She has taught mathematics at the high school and university levels. Her interests include curriculum development and professional development for teachers.